MVT, anti-derivatives and integration

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Problems

Problem 1. Two horses start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. (Hint: MVT.)

Solution: Let $s_1(t)$ and $s_2(t)$ be two functions that represent the position of the first and the second horse from the start at time t, respectively. Since both horses start at the same time, $s_1(0) = s_2(0) = 0$ (since they are both at the start at $t = 0$). Since they finish at a tie, this means it took them the same time t_0 to get to the finish, and so $s_1(t_0) = s_2(t_0)$. Consider the function $d(t) = s_1(t) - s_2(t)$. By MVT (assuming differentiability of both $s_1(t)$ and $s_2(t)$, of course) we get

$$\frac{d(t_0) - d(0)}{t_0 - 0} = d'(c) = s_1'(c) - s_2'(c)$$

for some c between 0 and t_0. But $d(t_0) = s_1(t_0) - s_2(t_0) = 0$ and $d(0) = s_1(0) - s_2(0) = 0$, so at c we have $s_1'(c) - s_2'(c) = 0$, i.e. $s_1'(c) = s_2'(c)$. This is exactly what we wanted, since the derivative of the distance is the velocity.

Problem 2. Let $f(x) = \frac{1}{x^2}$, and $F(x)$ be an antiderivative of f with the property $F(1) = 1$. **True or false:** $F(-1) = 3$.

Solution: False. Since $f(x)$ is not continuous on $\mathbb{R}$, we can pick different constants on $(-\infty, 0)$ and $(0, +\infty)$. In other words, an anti-derivative of $f(x)$ **does not have to be of the form** $F(x) = -\frac{1}{x} + C$. If it were, then $F(1) = -1 + C = 1$ so $C = 2$ would give $F(-1) = -\frac{1}{-1} + 2 = 3$. However, one can take, for example,

$$F(x) = \begin{cases} -\frac{1}{x} + 2 \\ -\frac{1}{x} + 100 \end{cases}$$

This is a perfectly good anti-derivative of $f(x)$ with $F(1) = 1$ but $F(-1) \neq 3$.

Problem 3. A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 minute later?

Solution: Acceleration is given by the derivative of the velocity, $a(t) = \frac{dv(t)}{dt}$. We are given $a(t) = \frac{dv(t)}{dt} = 20$ and so $v(t) = 20t + C$ for some constant C. Since at the time $t = 0$ the rocket is not moving, $v(0) = 0$, i.e. $C = 0$. This gives $v(t) = 20t$. In one minute, the speed will be $v(60) = 1200$ m/sec^2.

Problem 4. Compute the sum $\sum_{i=3}^{n} (i-2)^2$. You can use the formulas we've seen in class.

Solution: $\sum_{i=3}^{n} (i-2)^2 = 1^2 + 2^2 + \cdots + (n-2)^2 = \sum_{i=1}^{n-2} i^2 = \frac{(n-2)(n-1)(2n-3)}{6}$.

Problem 5. Compute $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{i(i+1)}$. (Hint: $\frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$)

Solution:

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

The last equality is just the result of cancellations: the whole sum "telescopes." Finally,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

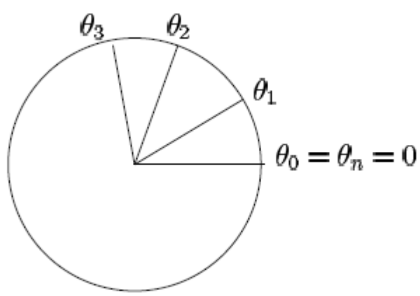
Problem 6. Compute the integral $\int_0^2 x dx$ **by definition**. Verify that this answer is the same as the usual (geometric) area under the graph of $f(x) = x$ over $[0, 2]$.

Solution: Because $f(x) = x$ is continuous, it is also integrable, and so it does not matter which Riemann sums to consider. For example, take the upper sum. Split $[0, 2]$ into n intervals of equal length $\frac{2}{n}$. Then i-th interval will be $\left[\frac{2(i-1)}{n}, \frac{2i}{n}\right]$. On this interval, the maximum of the function $f(x) = x$ is attained at the rightmost point $\frac{2i}{n}$, and this maximum value is $\frac{2i}{n}$. Thus, the upper sum is

$$U_n = \sum_{i=1}^{n} \frac{2i}{n} \cdot \frac{2}{n} = \frac{4}{n^2} \sum_{i=1}^{n} i = \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = \frac{2(n+1)}{n}$$

Thus, $\lim\limits_{n\to\infty} U_n = \lim\limits_{n\to\infty} \frac{2(n+1)}{n} = 2$. This coincides with the usual area of the triangle.

Problem 7. We cut a circular disk of radius r into n circular sectors, as shown in the figure, by marking the angles θ_i at which we make the cuts ($\theta_0 = \theta_n$ can be considered to be angle 0). A circular sector between two angles θ_i and θ_{i+1} has area $\frac{1}{2}r^2\Delta\theta$, where $\Delta\theta = \theta_{i+1} - \theta_i$.



We let $A_n = \sum\limits_{i=0}^{n-1} \frac{1}{2}r^2\Delta\theta_i$. Then the area of the disk, A, is given by

1. A_n, independent of how many sectors we cut the disk into.
2. $\lim\limits_{n\to\infty} A_n$.
3. $\int_0^{2\pi} \frac{1}{2}r^2 d\theta$.
4. all of the above.

Solution: all of the above. Clearly 2. and 3. are equivalent. But $\lim\limits_{n\to\infty} A_n = A_n$ since both equal simply to the area of the disk: the sum of the areas of sectors is the area of the whole disk.